Individual rationality and congestible good pricings

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Abstract: This paper studies the optimal pricings of a congestible good when users are privately informed both on the valuation of the good and on the disutility of congestion (i.e.: their unit waiting costs). Hence, conflicts appear between incentives and individual rationality. We show that the textbook price of a congestible good is higher than that of the same good when we suppose that the users’ valuations for that good are perfectly correlated with their unit waiting costs and lower in case the users have private unit waiting costs.

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1. Introduction

In a congested market, which is defined as a market for a product which value is affected by congestion, consumers’ utility decreases with the number of users demanding the same good. These users suffer from the externality of congestion. This negative externality characterizes the delay systems where the users share a fixed amount of capacity and receive a service at a quality that degrades with aggregate demand. In these systems, each user bears a delay cost that depends on the level of congestion and on his unit waiting cost.

The textbook congestion pricing which allows an efficient allocation shows that each user should pay a price that cover the marginal cost of production as well as the marginal cost of congestion imposed on the other users consuming the same good. The application of this pricing rule depends on the form of delay supported by users, which in turn depends on the technology and on the nature of service, and on the information that the seller has on each user’s valuation of the consumption of the good and the waiting costs.

We can mark out two kinds of delay. The first one consists in delivery delays due to queues. It concerns the industries in which demand is backlogged in queues for service such as a service industry in which each worker serves one customer at a time or the industries of the service sector that rent capacity or servers to customers.... Wilson [11] shows that the theory of priority service pricing constitutes an alternative for the spot pricing to achieve an efficient allocation. He determines the optimal mechanism specifying the price to be paid and the customer’s priority to obtain service when customers differ only in their valuation of service quality (that represents also their waiting costs) which is a function of the customer’s service order. He shows that each customer pays a charge sufficient to compensate lower-priority customers for the resulting degradation of the qualities of their services. More precisely, at each service order, a marginal saving in the price for later service equals the marginal waiting cost of the customer efficiently served in that order.\footnote{Moreover, Wilson [11] studies the operation of priority service in competitive markets. See also Reitman [9].}

The second kind of delay takes the form of a waiting time to satisfy the need of the user. This kind of delay concerns the industries in which the aggregate demand is served at the same time (all customers have the same service order) such as computer loads, electronic mail systems.... The time that an e-mail arrives at one’s destination and the duration for loading a file provide appropriate examples of this delay. This paper deals with this second aspect of delay.

For this kind of congestion, Mackie-Mason and Varian [5] determine the optimal pricing of a congestible network resources such as an ftp server, a router, a Web site, etc... in centrally planned, competitive, and monopolistic environments. They obtain the textbook congestion pricing: each user should use the system until the marginal benefit from his
usage equals the marginal cost of congestion that he imposes on other users.\textsuperscript{2} However, their analysis is based on the assumption that all users are equally harmed by congestion. This assumption means that they have the same unit waiting cost and it is supposed to be common knowledge. Even if this case can be viewed as a benchmark, it is a strong assumption because users do not really have the same waiting cost, i.e.: residential users and business users. So, they do neither bear the same cost nor impose the same cost of congestion. It follows that the seller is less informed than the users. But the seller is also less informed than the users on the valuation that they have on the consumption of the good. Hence, the seller faces a double asymmetry of information concerning both users’ valuations and users’ waiting costs.

We can therefore distinguish two cases.

1. The case where the valuation of the user is perfectly correlated to his waiting cost. It concerns congestible network goods that are mainly used by professional users because the more an agent values the good, the more he wants to avoid congestion. For example, some firms use the Internet to send huge databases or to communicate with their customers and subcontractors through live audio and video conferences.... These professional users have a high value for Internet and a high waiting cost.

2. The case, that is the most general, where the unit waiting cost is private information that differs from the user’s valuation. For example: the customers who only need to send e-mails and download small documents have high value for Internet but they do not have a high waiting cost. An adolescent has a high valuation for Internet because he can send more rapidly lovely messages than by the postal-mail, but he has not a high waiting cost compared with professional users.

It follows that the decision to consume the good depends closely on the case which we refer to. In technical terms, the individual rationality constraints of users differ. Our scope in this paper is two fold. Considering a congestible network good offered by a monopoly from a single capacity shared by some users, we determine, in a first step, the optimal pricing when users have a unit demand in order to compare, in a second step, the pricing levels with the benchmark.\textsuperscript{3,4}

\textsuperscript{2}Jebsi and Thomas \cite{1} study the implications of this result on the rent-efficiency trade-off when users have a several-unit demand. They show that the standard under-consumption with incomplete information no more holds whether the capacity of the network is exogenous or endogenous.

\textsuperscript{3}We consider a vertically integrated monopoly, that is, we omit the interdependence among congestible elements in the network.

\textsuperscript{4}Oren and al. \cite{8} propose a model of nonlinear pricing for both usage and capacity where the customers’ demand is represented by the load duration curve. They show that the capacity charges induce the buyers to truncate their purchase sets in order to reduce their charge for the capacity.
In the first case, the reader, familiar with the field of incentives theory, recognizes a problem of countervailing incentives. There is a conflict between user’s incentives and individual rationality.\textsuperscript{5} We show that the marginal cost of congestion that a user imposes on another user depends on the valuation of the marginal user who is indifferent about whether to consume the good or not. We find that the price with countervailing incentives is below the benchmark price. Indeed, the high delay cost born by users with high valuation for the good makes the outside opportunity more attractive which leads to a more elastic demand. It obliges the seller to decrease the price of the good.

In the second case, the reader will recognize a problem of random participation. There is now a conflict between user’s incentives and probabilistic individual rationality.\textsuperscript{6} We show in this case that the marginal cost of congestion imposed by a user to another is the average of marginal delay costs of users consuming the good. We show in this case that the price fixed by the seller is above the benchmark price.

Hence, we show that the level of the price fixed by the seller in the general case (second case) is the highest. However, the level of the price with countervailing incentives is the lowest. The textbook price has thus an intermediary level.

The paper is organized as follows: in section 2, we describe the model; in section 3, we determine the congestion pricing that allows an efficient use of the network resource and we compare the prices levels of the two last cases with the benchmark one.

2. The model

A monopoly sells a good on a congestible network to a continuum of consumers. Users have a unit demand. Let \( v \in [\underline{v}, \overline{v}] \) be the valuation for the good. If we note \( q \) the probability to consume the good and \( p \) the monetary transfer to get it, the utility of consumer is:

\[
u = \underline{v}q - p\]

The seller does not observe the valuation \( v \), but it has prior beliefs on the valuation noted \( f(.) > 0 \) on \([\underline{v}, \overline{v}]\), with cumulative \( F(.) \). We make the following assumption:

\textbf{A1:} \( f(.)/(1 - F(.)) \) is increasing, which is satisfied by most of the usual distributions.

The monopoly offers a mechanism \((p(\hat{v}), q(\hat{v}))\) specifying the price to be paid and the probability to consume the good as a function of the user’s report \( \hat{v} \). Without loss of generality, we assume that the service supply cost is zero. The revenue of the seller is:

\[
R = \int_{\underline{v}}^{\overline{v}} p(v)q(v)f(v)dv
\]

To maximize his revenue, the seller has to respect the feasibility constraints (see Maskin and Riley [7]). Incentives constraint ensures that users report their true valuation. We

\textsuperscript{5}See Maggi and Rodriguez-Clare [6]; Lewis and Sappington [4] for more details. See also Jullien [3].

\textsuperscript{6}See Rochet and Stole [10] for a complete analysis.
must have:
\[ u(v) = vq(v) - p(v) \geq vq(\hat{v}) - p(\hat{v}), \forall v, \hat{v} \in [v, \overline{v}] \]

After standard manipulations, we can show that this constraint is satisfied if for two valuations \( v \) and \( \hat{v} \) that have the same probability to get the good (i.e.: \( q(v) = q(\hat{v}) \)) the users must pay the same monetary transfer: \( p(v) = p(\hat{v}) = p \). Since each consumer faces a take-it or leave-it offer, the revenue becomes \( pQ(p) \), where \( Q(p) = \int q(v)f(v)dv \) is the aggregate demand. The maximum revenue requires:
\[
p = \frac{Q(p)}{Q'(p)} \tag{2.1}
\]

The optimal price results from the following trade-off. By increasing the price \( p \) by \( dp \) the seller increases his revenue on the aggregate demand by \( Q(p)dp \), but this induces a loss of users whose valuations are close to \( p \), i.e. \( Q'(p)dp \) since such users can no more consume the good.

The aggregate demand depends on the proportion of consumers that participates in the offered mechanism. For a given network capacity \( K > 0 \), the decision to consume the good or not depends on the level of congestion faced by the user since this leads to a congestion related disutility. Since the capacity is shared by users, the congestion function giving the level of congestion is: \( Y(p) = \frac{Q(p)}{K} \).\(^7\) In what follows we consider a linear delay cost:
\[
\theta Y(p)
\]
where \( \theta \) is the unit waiting cost (or also the willingness to pay in order to avoid congestion ).

We deduce that consumer with type \( v \) participates if:
\[
u(v) - \theta Y(p) \geq 0
\]
where 0 denotes the value of user’s outside opportunity.

Before determining the optimal pricing schedules, we make the following assumption:
\[\textbf{A2} : K \text{ is high.}\]

This second assumption implies that the users suffer from congestion even if its level is not too high. This is verified when we consider the second aspect of delay presented in the introduction. It allows us to make comparisons between the different configurations.

### 3. Optimal pricings for congestion

In the following analysis, we distinguish three possible cases:

\(^7\)See Reitman [9] for the formulation of congestion functions of different forms of congestion in particular for the processor sharing.
• users have a common knowledge of the same unit waiting cost, i.e: $\theta = \theta_0$. This reflects the standard assumption in the textbook and can be viewed as a benchmark;

• users have a unit waiting cost corresponding to their valuation of the good, i.e.: $\theta = v$. This case reflects the idea that the network concerns most professional users;

• users have a private unit waiting cost, i.e.: $\theta \neq v$. Then $\theta$ is a random variable with density $l(.) > 0$ on $\Theta \subset \mathbb{R}^+$. This case is the most general, and the two previous cases can be viewed as special cases of this one.

3.1. The unit waiting cost is a common value

In this case, all users support the disutility of congestion $\theta_0 Y(p)$. So a user consumes the good if:

$$u(v) - \theta_0 Y(p) \geq 0$$

$$\Leftrightarrow v \geq p + \theta_0 Y(p)$$

It follows that high value users consume whereas low value users do not. The aggregate demand becomes:

$$Q(p) = 1 - F(p + \theta_0 Y(p))$$

(3.1)

The revenue is then:

$$R_1(p) = p(1 - F(p + \theta_0 Y(p)))$$

PROPOSITION 1. With a common unit waiting cost, the optimal congestion pricing is:

$$p_1 = \frac{1 - F(p_1 + \theta_0 Y(p_1))}{f(p_1 + \theta_0 Y(p_1)) + (1 - F(p_1 + \theta_0 Y(p_1)))\frac{\theta_0}{K}}$$

PROOF. After differentiation and collection of terms, we obtain:

$$Q'(p) = - \frac{f(p + \theta_0 Y(p))}{1 + f(p + \theta_0 Y(p)) \theta_0 \frac{1}{K}}$$

(3.2)

Inserting (3.1) and (3.2) in equation (2.1) gives the proposition 1. Under the assumption A1, the revenue is quasi-concave, so the policy is optimal (see appendix 1). ■

The price is composed of two parts: the first one has the following interpretation. By increasing the price $p$ by $dp$ the seller increases his revenue on the aggregate demand by $(1 - F(p + \theta_0 Y(p)))dp$, but this induces a loss of users whose valuation are close to $p$, i.e. $f(p + \theta_0 Y(p))dp$ since such users can no more consume the good.

The second one is the so called marginal cost of congestion that a user imposes on the other users consuming the good. It corresponds to the product of the marginal cost of congestion that a user imposes on another, $\frac{\theta_0}{K}$, by the number of users $(1 - F(p_1 + \theta_0 Y(p_1)))$. 
When we assume that the waiting cost of users is common, we get the traditional textbook congestion pricing which allows the efficient use of the resource and shows that each user should use the system until the marginal benefit from his usage equals the marginal cost of congestion that he imposes on the other users.

3.2. The unit waiting cost corresponds to the valuation of the good

In this case, users support a disutility of congestion of \( vY(p) \). Thus, a user with type \( v \) consumes if:

\[
\begin{align*}
    u(v) - vY(p) & \geq 0 \\
    \iff v & \geq p + vY(p)
\end{align*}
\]

So, for a given price \( p \), the more users value the good, the more they suffer from a high disutility of congestion. Clearly, countervailing incentives problem will be inevitable. But under assumption A2, the level of congestion is weak enough, so that \( 1 - Y(p) > 0 \). So, the net utility of consumers \( u(v) - vY(p) \) is increasing in \( v \).\(^8\) Let us define \( v_m \) as

\[
v_m = p + v_mY(p)
\]

that is, as the marginal user who is indifferent about whether consuming or not the good. It follows that the types who are slightly above \( v_m \) consume whereas those who are below \( v_m \) do not. Thus, as in the previous case, only high value users consume the good. The aggregate demand becomes:

\[
    Q(p) = 1 - F(p + v_mY(p)) \tag{3.3}
\]

The revenue is then:

\[
    R_2(p) = p(1 - F(p + v_mY(p)))
\]

In this case:

**Proposition 2.** With a unit waiting cost corresponding to the valuation, the optimal congestion pricing is:

\[
    p_2 = \frac{1 - F(p_2 + v_mY(p_2))}{f(p_2 + v_mY(p_2))}(1 - Y(p_2)) + (1 - F(p_2 + v_mY(p_2)))\frac{v_m}{K}
\]

**Proof.** We show in appendix 2, that

\[
    Q'(p) = -\frac{f(p + v_mY(p))}{1 - Y(p) + f(p + v_mY(p))\frac{v_m}{K}} \tag{3.4}
\]

Inserting (3.3) and (3.4) in equation (2.1) completes the proof. ■

\(^8\)Jebsi and Thomas [2] study also the cases where the network capacity level is low and intermediary.
We obtain the same pricing principle as that of the benchmark. The two terms of \( p_2 \) have fundamentally the same interpretations. However, two interesting features appear.

First, the set of marginal users \( f(\cdot) \) is weighted by \( 1/(1 - Y(\cdot)) \). Indeed, under countervailing incentives, the marginal user depends on the price \( p_2 \) which the seller must take into account to determine the loss of users whose valuation are close to \( p_2 \), who can no more consume the good. The set of marginal users is \( f(\cdot)/(1 - Y(\cdot)) \). Then, countervailing incentives influence the price to be paid by modifying the set of marginal users.

Second, although the users have different unit waiting costs, the marginal cost of congestion that a user \( v \) imposes on another is \( v_m/K \), which is the marginal delay cost of the marginal user. This is the lowest cost because \( v_m \) is the lowest user’s valuation that consumes the good. So, the marginal cost of congestion that a user imposes on others is \( (1 - F(p_2 + v_m Y(p_2))) v_m K \).

In the next proposition, we determine the countervailing incentives effect on the price level in reference to the benchmark case.

**Proposition 3.** Assume \( v_m = \theta_0 \), then \( p_2 < p_1 \).

**Proof.** Let: \( h(p, v_m) = p + v_m Y(p) \). We have:

\[
\frac{1 - F(h(p_2, v_m))}{f(h(p_2, v_m))} (1 - Y(p_2)) + (1 - F(h(p_2, v_m))) \frac{v_m}{K} < \frac{1 - F(h(p_2, v_m))}{f(h(p_2, v_m))} + (1 - F(h(p_2, v_m))) \frac{v_m}{K} \quad (3.5)
\]

If \( v_m = \theta_0 \), (3.5) is rewritten:

\[
p_2 < \frac{1 - F(h(p_2, \theta_0))}{f(h(p_2, \theta_0))} + (1 - F(h(p_2, \theta_0))) \frac{\theta_0}{K} \quad \Leftrightarrow \quad 0 < -p_2 \frac{f(h(p_2, \theta_0))}{1 - F(h(p_2, \theta_0))} + 1 + f(h(p_2, \theta_0)) \frac{\theta_0}{K}
\]

Since \( R_1(\cdot) \) is quasi-concave, we find that \( p_2 < p_1 \). ■

Under the assumption that the unit waiting cost of the marginal user \( v_m \) is exactly equal to the common waiting cost \( \theta_0 \), we find that the price with countervailing incentives is lower than that of the benchmark.

The users who are valuating the more the good are also those suffering the more from congestion. Then, the high delay cost born by these users makes the outside opportunity more attractive. This conflict leads to a more elastic demand and it forces the seller to decrease the price of the good.
3.3. **The unit waiting cost is a private value**

In this case, each consumer has his own waiting cost and supports a disutility of congestion of $\theta Y(p)$. A user consumes the good if:

$$u(v) - \theta Y(p) \geq 0 \iff v \geq p + \theta Y(p)$$

Since $\theta$ is unknown to the seller, there is no marginal user in this case. However, he knows that there is a proportion $l(\theta)$ of users with waiting cost $\theta$. So the aggregate demand becomes:

$$Q(p) = \int_{\theta} \left(1 - F(p + \theta Y(p))\right)l(\theta)d\theta = E\left(1 - F(p + \theta Y(p))\right)$$

(3.6)

where $E$ denotes the expected operator over $\theta$.

The revenue becomes:

$$R_3(p) = pE(1 - F(p + \theta Y(p)))$$

**Proposition 4.** With a private unit waiting cost, the optimal congestion pricing is:

$$p_3 = \frac{E(1 - F(p_3 + \theta Y(p_3)))}{Ef(p_3 + \theta Y(p_3))} + E\left[1 - F(p_3 + \theta Y(p_3))\right] \frac{Ef(p_3 + \theta Y(p_3))\theta}{Ef(p_3 + \theta Y(p_3))}$$

**Proof.** We have, after differentiation and collection of terms:

$$Q'(p) = -\frac{Ef(p + \theta Y(p))}{1 + Ef(p + \theta Y(p))}\frac{\theta}{Ef(p + \theta Y(p))}$$

(3.7)

Then, inserting (3.6) and (3.7) in equation (2.1) gives the proposition 4.

Again, we obtain the same structure of pricing as in the benchmark case, but each term of the price is now evaluated by the expected operator over $\theta$.

However, since the users have different waiting costs, the marginal cost that a user imposes on another is given by $\frac{Ef(p_3 + \theta Y(p_3))\theta}{Ef(p_3 + \theta Y(p_3))}$. This cost is the average of marginal delay costs of users consuming the good. Therefore, the marginal cost of congestion that a $\theta$ user imposes on others is the product of $E\left[1 - F(p_3 + \theta Y(p_3))\right]$ by $\frac{Ef(p_3 + \theta Y(p_3))\theta}{Ef(p_3 + \theta Y(p_3))}$.

It follows that a new insight appears: random participation influences the management of congestion. Indeed, managing congestion implies now replacing the "standard" marginal cost of congestion by the average marginal cost of congestion. Thus, the marginal cost of congestion can be viewed as a polar case of congestion management where users do not really have private information on their waiting costs, as in the previous cases.
Before comparing \( p_1 \) and \( p_3 \) we make the following assumption:

\[ \text{A3: } -p \frac{f(.)}{1-F(.)} + \frac{f(.) \theta}{K} \text{ is convex.} \]

It can be noted that most of the standard distributions are such that \( \frac{f(.)}{1-F(.)} \) is concave. Combined with A2, this assumption is less restrictive than it can appear.

**Proposition 5.** Assume \( E(\theta) = \theta_0 \), then \( p_1 < p_3 \).

**Proof.** See appendix 3. ■

If the average unit waiting cost \( E(\theta) \) is exactly equal to the common unit waiting cost \( \theta_0 \), we find that the benchmark price is weaker than the price with random participation.

Indeed, when the waiting cost is private information, the seller by increasing his price by \( dp \) increases his revenue by \( E(1-F(p+\theta Y(p)))dp \). But, he loses the revenue \( E f(p+\theta Y(p))dp \). Assumption A3 ensures that the expected losses are weaker than the expected gains when they are evaluated at \( p_1 \). More precisely, A3 implies that when the unit waiting cost is unknown the demand is less elastic than the demand with a common unit waiting cost. Therefore the seller can apply a greater mark-up.

### 4. Conclusion

We have determined in this paper the optimal pricings of a congestible network good when users bear a delay cost. The seller faces double asymmetry of information because the users have different valuations for the good and different waiting costs. We have showed that these asymmetries of information influence the textbook congestion pricing.

When the valuation’s user is perfectly correlated to his unit waiting cost, we show that the set of marginal users is affected by the countervailing incentives and the marginal cost of congestion depends on the valuation of the marginal user.

In the general case where users have waiting costs that differ from their evaluations, the textbook pricing is modified because in this case, we show that the management of congestion consists in taking into account the average marginal cost of congestion.

The essential limit of our model is that in the three cases treated, each user imposes on other users the same marginal cost of congestion. The reason is that it has been supposed that each user demands a single unit of the good. So, the natural extension of this paper is to treat the case where users have a several-unit demand.

### 5. Appendixes

#### 5.1. Appendix 1

Let \( g(p) = p + \theta_0 Y(p) \). It follows that the revenue can be rewritten:

\[ R_1(p) = p(1 - F(g(p))) \]
So

\[ R_1'(p) = -p \frac{f(g(p))}{1 + f(g(p))\theta_0 K} + (1 - F(g(p))) \]

For the optimal price \( p_1 \), we obtain:

\[ R_1'(p_1) = -p_1 \frac{f(g(p_1))}{1 + f(g(p_1))\theta_0 K} + (1 - F(g(p_1))) = 0 \]

\[ \Leftrightarrow -p_1 \frac{f(g(p_1))}{1 - F(g(p_1))} + 1 + f(g(p_1))\theta_0 K = 0 \]

So, \( R_1''(p_1) \) is equal to:

\[ \frac{-f(g(p_1))}{1 - F(g(p_1))} - p_1 \left( \frac{f(g(p_1))}{1 - F(g(p_1))} \right)' + f'(g(p_1))g'(p_1)\theta_0 K = 0 \]

\[ = -\frac{f(g(p_1))}{1 - F(g(p_1))} - g'(p_1) \left[ \frac{f'(g(p_1))}{1 - F(g(p_1))} \left( p_1 - F(g(p_1))\frac{\theta_0 K}{f(g(p_1))} \right) + \frac{p_1 f(g(p_1))^2}{(1 - F(g(p_1)))^2} \right] \]

But:

\[ p_1 > p_1 - (1 - F(g(p_1)))\frac{\theta_0 K}{f(g(p_1))} = \frac{1 - F(g(p_1))}{f(g(p_1))} > 0 \]

where the equality follows from proposition 1. Since A1 implies for all \( p \), \( \frac{pf'(\cdot)}{1 - F(\cdot)} + \frac{pf(\cdot)^2}{(1 - F(\cdot))^2} > 0 \), we find that \( R_1''(p_1) < 0 \) since

\[ g'(p_1) = 1 + \frac{\theta_0 K}{f(g(p_1))} > 0 \]

5.2. Appendix 2

\[ Q(p) = 1 - F(p + v_m Y(p)) \]

We have:

\[ Q'(p) = -f(p + v_m Y(p)) \left[ 1 + \frac{dv_m}{dp}Y(p) + v_m \frac{Q'(p)}{K} \right] \]

Using the definition of marginal users \( v_m \), we get:

\[ \frac{dv_m}{dp} = \frac{1 + \frac{v_m K}{Q'(p)}}{1 - Y(p)} \]

So,

\[ Q'(p) \left[ 1 + f(p + v_m Y(p))\frac{v_m K}{1 + \frac{Y(p)}{1 - Y(p)}} \right] \]

\[ = -f(p + v_m Y(p)) \left[ 1 + \frac{Y(p)}{1 - Y(p)} \right] \]
By simplifying the terms $1 + \frac{Y(p)}{1 - Y(p)}$, we obtain:

$$Q'(p) = -\frac{f(p + v_m Y(p))}{1 - Y(p) + f(p + v_m Y(p)) \theta K}$$

### 5.3. Appendix 3

Let $h(p, \theta) = p + \theta Y(p)$, we obtain

$$R_3'(p) = -p \frac{Ef(h(p, \theta))}{E(1 + f(h(p, \theta)) \frac{\theta}{K})} + E(1 - F(h(p, \theta)))$$

Moreover, we have

$$Ef(h(p, \theta)) = E \left[ \frac{f(h(p, \theta))}{1 - F(h(p, \theta))} (1 - F(h(p, \theta))) \right]$$

$$< E \left[ \frac{f(h(p, \theta))}{1 - F(h(p, \theta))} \right] E \left[ (1 - F(h(p, \theta))) \right]$$

since $\frac{f(.)}{1 - F(.)}$ and $1 - F(.)$ negatively covariate under A1.

The inequality can be rewritten

$$-pEf(h(p, \theta)) + E(1 - F(h(p, \theta)))E(1 + f(h(p, \theta)) \frac{\theta}{K})$$

$$> -pE \left[ \frac{f(h(p, \theta))}{1 - F(h(p, \theta))} \right] E(1 - F(h(p, \theta))) + E(1 - F(h(p, \theta)))E(1 + f(h(p, \theta)) \frac{\theta}{K})$$

$$\iff \left[ -p \frac{Ef(h(p, \theta))}{E(1 + f(h(p, \theta)) \frac{\theta}{K})} + E(1 - F(h(p, \theta))) \right]$$

$$> \frac{E(1 - F(h(p, \theta)))}{E(1 + f(h(p, \theta)) \frac{\theta}{K})} \left\{ -pE \left[ \frac{f(h(p, \theta))}{1 - F(h(p, \theta))} \right] + E(1 + f(h(p, \theta)) \frac{\theta}{K}) \right\}$$

$$\iff R_3'(p) > \frac{E(1 - F(h(p, \theta)))}{E(1 + f(h(p, \theta)) \frac{\theta}{K})} \left\{ E \left[ -p \frac{f(h(p, \theta))}{1 - F(h(p, \theta))} + 1 + f(h(p, \theta)) \frac{\theta}{K} \right] \right\}$$

Under A3, the function in the brackets is convex. Using Jensen’s inequality and $h_\theta(p, \theta) = Y(p)$, we have:

$$E \left[ -p \frac{f(h(p, \theta))}{1 - F(h(p, \theta))} + 1 + f(h(p, \theta)) \frac{\theta}{K} \right] > \left[ -p \frac{f(h(p, \theta_0))}{1 - F(h(p, \theta_0))} + 1 + f(h(p, \theta_0)) \frac{\theta_0}{K} \right]$$

since $E(\theta) = \theta_0$. Combining the two last inequalities, we obtain at $p_1$

$$R_3'(p_1) > \frac{E(1 - F(h(p_1, \theta_0)))}{E(1 + f(h(p_1, \theta_0)) \frac{\theta}{K})} \left[ -p_1 \frac{f(h(p_1, \theta_0))}{1 - F(h(p_1, \theta_0))} + 1 + f(h(p_1, \theta_0)) \frac{\theta_0}{K} \right] = 0$$

where the equality follows from proposition 1. Since the revenue is quasi-concave, we have $p_3 > p_1$. 
Bibliography


