Interbanking Networks: Towards a Small Financial World?
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Abstract
In a standard stylised frame derived from Diamond Dybvig, banks operate within a network of debt contracts. Working in network enables banks to decentralize a Pareto Optimal allocation while it is impossible if banks operate in isolation. However, this outcome depends on the architecture of the network which itself depends on network participant number and on the cost structure. In a general frame with no cost, two network structures only decentralize first best outcome. The first structure highlights a Small World property as banks must be bound together at a network distance that equals at maximum 2. The second structure exhibits a strict regular topology. The rise in the number of competing banks leads to a more than proportional rise in interbank lending operations. In a frame with positive cost, we prove a single architecture both minimizes aggregate costs and decentralizes first best outcome. However, this topology has little chance to emerge as it exhibits unbalanced cost sharing among palyers. Aggregate cost efficiency is indeed not compatible with individual cost minimization.

Keywords: network, bank, debt, financial stability.
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1 Introduction

Financial distress and a propagation channel are both needed for contagious bank failures to occur. If the first aspect has been highly scrutinized by economic theory, the second one lacks of a convenient framework to analyze financial or interbanking architecture. The way interbank architecture is organized is indeed essential to understand current banking business and particularly banking and systemic risk.

What we mean by interbanking architecture refers at least to two different aspects: first the interbank debt market, and second the private interbank payment and clearing system. On the one side, interbank debt exchange plays a crucial role by channeling liquidity from institutions with a surplus of funds to those in need. Central Bank is not indeed the single liquidity provider to the financial system. In the European Union for e.g., ECB Open Market operations are not continuous, they occur twice a week along a fixed calendar. When central bank does not intervene, a completely private system of interbank short term loans enable financial institutions to exchange funds. On the other side, a significant payment and clearing system has emerged outside the control of banking regulators and outside any Central Bank intervention trough which banks daily exchange huge amounts of liquidity. For e.g. the CHIPS network in the US functions as a private self-regulating interbank clearinghouse among financial institutions. SWIFT system is some kind similar in Europe to enable banks to directly compensate payments and exchange short term loans.

Consequently, banks are embedded within a network of financial relations which expands at the rhythm of financial globalization and liberalization. By interbank network we will refer to the development of financial flows among banks such as interbank loans, any payment flows or contracts between two financial institutions implying fund exchanges.

Determining how the network of interbank exchange of funds is and should be structured is a basic question which remains unsolved. To be accurate, such a topic has to be studied as follows:

- first, what are the gains for a bank to work in networks;
- second which interbank architectures enable to maximize the gains of the network;

Quantitatively, further to financial liberalization, the number of financial institutions carrying out interbank credit operations and the volumes of these operations have boomed. Those features have been particularly well documented by Furfine (2001 & 1999), or Bernard & Bisignano (2000). Rochet & Tirole (1996b) for example note “Over the past twenty years, the growing integration of financial markets, the development of new financial instruments, and the advance of computer technology have all contributed to a remarkable growth of financial activity in the main industrialized countries. One of the most significant consequence of this growth has been the unprecedented increase in the volume of trade on the large value interbank payment systems which itself has resulted in a massive increase in intra-day overdrafts on those systems”. Documenting this fact, Rochet and Tirole cite figures extracted from Goodheart & Schoenmaker (1993). In the US, the daily volume of interbank flows amounts to USD bn 800. Consequently, banks are embedded within a continuously expanding network of financial relations.
-third, how interbank network architecture influence financial stability and an cost efficiency;

-fourth, what is the theoretic foundation to the changes in actual interbanking architecture.

To answer these questions we shall consider a standard stylized frame derived from Diamond Dybvig, were banks are operating within a network of debt contracts. Our main findings are the following. Working in networks is a way for bank to improve depositors’ welfare and to decentralize a Pareto Optimal allocation while it is impossible if banks operate in isolation. However, this outcome depends on the architecture of the network which itself depends on network participant number and on the cost structure. In the $2p$ region frame with no-cost ($p$ being a natural integer), interbank networks which decentralize first best are either star-shaped or networks in which each bank has at least $p$ partners. A wide range of banking structures are not efficient. To keep the first best decentralization property and improve depositors welfare, a minimum density of the network is necessary. Banks must be bound together at a network distance that equals at maximum 2, which highlights a Small World property, meaning that banks are separated one from the other only by a very short chain of intermediaries. The rise in the number of competing banks which characterize financial and banking liberalization thus leads to a rise in the number of links between banks, and a rise in interbank lending operations at a rhythm more than proportional to the number of banks. Total number of edges is indeed a quadratic function of the number of players. The higher the number of banks, the denser are interbank lending relations. Finally, a single architecture both minimize aggregate costs and decentralize first best outcome. But this topology has little chance to emerge as cost sharing is then unbalanced among players. Aggregate cost efficiency is not compatible with individual cost minimization. These theoretical results are comforted by the empirical study carried out by Boss and alii (2003) on the Austrian interbank market.

Despite this undisputable rise as the autonomous financial players, very little attention has been devoted to interbank networks even in the considerable literature that developed on banking and banking risk management following Diamond and Dybvig’s seminal work\textsuperscript{2}.

\textsuperscript{2}The first models to be developed are indeed centered on single bank fragility. The canonical setup established by Diamond and Dybvig (1983) highlights an intrinsic banking fragility stemming from the role played by the bank as a maturity transformer. Any bank is indeed financed by short term deposits, while it finances long term investments while lending to the economy. This unmatched asset / liability maturity structure may expose banks to self-fulfilling bank-runs.

The sunspot run equilibrium hypothesis, made by Diamond and Dybvig, has been the first one to be dropped and replaced by the assumption that runs are consecutive to the release of new information about the viability of bank’s investments. Main models under this hypothesis are Gorton (1985), Jacklin and Battacharya (1988), or Chari and Jaghanathan (1988). Based on an information those runs are ”efficient” while the one highlighted by Diamond and Dybvig are not. Extensions of the bank run literature to multiple bank systems begins with Garber and Grilli (1989). In their model contagion among banks is channeled through income effect experienced by depositors. De Bandt (1985) and Temzelides (1997) rely both on a revision of depositors expectations on the expected return on bank’s asset. The expectation revision
In fact, interbank market exposure appears only with Rochet & Tirole (1996a) where the authors raise the problem of moral hazard induced by lending / borrowing operations on the interbank market.

Even in the multiple bank case, models do not study the way the banking system is structured and how it influences financial stability. When contagion occurs among banks, the models -except for the following two- in fact assume that financial institutions are linked through financial line chains.

The contributions of Allen & Gale (2000) and Freixas, Parigi & Rochet (2000) are the first ones to introduce different shapes for the banking structure. Taking into account the way banking systems are structured to study banking issues thus appears to be a recent improvement in models of banking fragility. However, the previous contributions remain incomplete.

First, they only study some of the possible topologies and not all of them.

Second, the issue of stability is studied under the particular angle of contagion which can be defined as the propagation of an isolated financial event -such as an isolated run- through a network of links. The concept of stability has to be larger to determine which networks are ex-ante possible. Stability refers to a general case with no bankruptcy.

Third, they do not explore the link between participant number and network topology.

Network theory concepts are of interest to enrich financial literature to determine the topology of these networks and the way this topology emerges.

Indeed, a recent and rapidly growing literature on network formation has developed since a few years. Based on strong theoretic tools, its main field of applications are information about job opportunities, trade and exchange of goods in non-centralized markets, R&D and collusive alliance among firms or international alliances and trading agreements.

The plan is as follows. Section 2 exposes the allocation obtained when depositors have access to an autarkic bank and compares it to Central planner's allocation. Section 3 studies decentralization of first best outcome through an interbank network. Section 4 considers network architectures with no cost. Section 5 introduces a positive cost to be paid by each bank to partake in the

3Their paper show that peer monitoring among banks is an efficient way of tackling this issue. The model is however very sensitive to parameters value, and the banking system can be driven to bankruptcy for a small increase in the liquidity shock occurring in the first period.


5Granovetter (1973), Calvo-Armengol (2000) or Ioannides & Datcher Loury (2002) for example


8Goyal & Joshi (2001) or Furusawa & Konishi (2002)
network in the 4 bank and the $2p$-bank case. Section 6 concludes.

2 The Model

This section describes a simple model adapted from Allen and Gale (2000). Within this frame, the interbank network enables an optimal risk sharing while liquidity shocks are stochastic.

Time is discrete and divided into 3 periods $t = 0, 1, 2$. There is a single available and storable good (numeraire) which can be either consumed or invested. At $t = 0$, investment can be made in two different assets only. The first one is a liquid short term storage technology. One unit of good invested in this technology at date $t$ produces one unit at date $t + 1$. The second one is a long term illiquid risky asset. Investment in this asset can only be made at the first date. One unit invested at $t = 0$ produces $R > 1$ units of good at period 2. Illiquidity is not complete. Should the long term asset be sold at the intermediate date, thus before it matures, it produces only $r$ units of consumption good, with $0 < r < 1$. Let us assume that the early liquidation is a physical depreciation, the liquidation value being treated as the constant ”scrap value” of investment.

Economy is constituted with $2p$ ex-ante identical regions. $p$ is a strictly positive natural integer. We note $V_{2p} = \{i_1, i_2, \ldots, i_{2p}\}$ the set of regions constituting the economy. In each one operates a single monopolistic bank, each bank is a network vertex. This regional structure can be interpreted in different ways. It can be considered as geographical areas for instance, or specialized branches in the banking industry.

Each region is populated with a continuum of ex-ante identical depositors. Each depositor is endowed with one unit of consumption at $t = 0$ and receives no further endowment. Consumers have standard Diamond Dybvig preferences. With probability $\lambda$ they are early consumers who value only date 1 consumption. With probability $1 - \lambda$ they are late consumers who value only date 2 consumption. Consumers preferences are thus given by

$$U(C_1, C_2) = \begin{cases} U(C_1) \text{ with probability } \lambda \\ U(C_2) \text{ with probability } 1 - \lambda \end{cases}$$

$C_t$ being consumption at date $t = 0, 1, 2$. $U$ is twice continuously differentiable, growing and strictly concave.

Probability $\lambda$ varies across regions. $\lambda^i$ is the probability to be an early consumer in region $i$. There are two possible values for $\lambda^i$: a high value, noted $\lambda^i_h$, and a low value noted $\lambda^i_l$, with $0 < \lambda^i_l < \lambda^i_h < 1$. The value taken by $\lambda$ depends on the state of nature, $S_i$. In each state of nature, half of the regions are hit by a high liquidity shock and the other half by a low liquidity shock. The realization of the liquidity shock in the 4 region case is given in the following table:
Each region has the same ex-ante probability to face a high demand for liquidity equals to a half. Aggregate demand for liquidity is known with probability 1 ex-ante, the distribution of those shocks is the single unknown variable.

All uncertainty is resolved at \( t = 1 \), when the state of nature is revealed. At time \( t = 1 \) each consumer knows his type, this is a private unobservable information. This characteristic is essential to the realization of bank runs which happen only when patient consumers do not reveal their true type and act as if they where short term consumers. As it is standard in such a model frame à la Diamond Dybvig, it is optimal for consumers to act cooperatively and to create a mutual bank that enables them to self insure against the risk of being type 1 (early consumers).

We shall study two benchmarks: the autarkic and the Central planner solutions.

### 2.1 The Autarkic Solution

The bank then maximizes the consumer expected utility, subject to liquidity constraints. The amount redeemed by agents at date \( t = 1 \) is the realization of the liquidity risk of each agent. We assume the liquidity risk is highly volatile\(^9\). We shall assume thus the low demand for liquidity to be small, and the high demand to be very large such that we get \( \lambda_l \ll \lambda_h \). Let us note \( \gamma = \frac{\lambda_h + \lambda_l}{2} \), the average liquidity shock. A bank in autarky located in region \( i \) maximizes the ex-ante expected utility of the representative consumer

\[
Max_{b; k_i} \left[ \frac{1}{2} \lambda_h U(C_{i1}) + (1 - \lambda_h) U(C_{i2}) \right] + \frac{1}{2} \left[ \lambda_l U(C_{i1}) + (1 - \lambda_l) U(C_{i2}) \right]
\]

Subject to the following constraints

\[
b^i + k^i \leq 1 \tag{2}
\]
\[
\lambda_h C_{i1}^i \leq b \tag{3}
\]
\[
(1 - \gamma)C_{i2}^i \leq Rk \tag{4}
\]
\[
C_{i1}^i \leq C_{i2}^i \tag{5}
\]

At \( t = 0 \), bank chooses a portfolio \((k^i, b^i) \geq 0\) where \( k \) is the per depositor amount invested in the long term asset (capital), while \( b \) is the per depositor

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\(^{9}\)The reason for this volatility are however not modeled here. This is just a formal assumption to prevent that bank with low liquidity reserves are not able to cope with a high demand for liquidity.
amount invested in the short term asset (bonds), (Inequality 1). Inequality (2) is date 1 feasibility constraint, while date 2 feasibility constraint is given by inequality (3). Each bank has to abide by high demand for liquidity constraint at \( t = 1 \) because of the assumptions of the model in this case which make it impossible to borrow from any other institution (autarky) and because early liquidation is so costly that it is useless to get liquidity. However, at \( t = 2 \), the bank’s resources have to cover the payment towards late depositors which are in proportion \( \lambda_h^i \) with probability \( \frac{1}{2} \) or \( \lambda_h^i \) with probability \( \frac{1}{2} \) depending on \( t = 1 \) realized shock. Inequality (4) is the incentive constraint.

First order conditions give us the following solutions

\[
\lambda_h^i \hat{C}_i^1 = \hat{b}^i \tag{6}
\]

\[
(1 - \lambda_h^i)\hat{C}_i^2 = R\hat{k}^i \tag{7}
\]

\[
U'(\hat{C}_i^1) = R \frac{\lambda_h^i}{\gamma} U'(\hat{C}_i^2) \tag{8}
\]

With \( \frac{\lambda_h^i}{\gamma} > 1 \)

As standard in models a la Diamond Dybvig, short term depositors benefit from the mutual bank. However, in autarky, a bank has to keep a high amount of liquid reserves to prevent itself from the risk of facing a high demand for liquidity, and consequently being driven to bankruptcy. The amount invested in the long term asset is relatively small.

### 2.2 The Central Planner Solution

Central planner has the ability to transfer liquidity across regions at \( t = 1 \), once the liquidity shock distribution is known but before depositors’ withdrawals. There is no aggregate uncertainty. So, he has only to abide by average liquidity constraints. Consequently the allocation does not depend on the state of nature. All depositors are treated alike (they are all ex-ante identical). Each early consumer, receives \( C_1 \) and each late consumer \( C_2 \) whatever the bank in which he deposits his endowment, and whichever the state of nature may be. All banks face symmetric ex-ante situations with a \( \frac{1}{2} \) probability to be hit by a high shock. Moreover they have the same technical abilities and they have access to the very same assets. As the program is solved for a representative region, we shall omit region’s index.

\( t = 0 \) constraint remains unchanged, i.e.

\[
k + b \leq 1 \tag{9}
\]

The payoffs make it optimal for the bank to pay for short term withdrawals by bond selling with \( \gamma \) being the average date 2 liquidity need \( (\gamma = \frac{\lambda_h^i + \lambda^i}{2}) \).

\( t = 1 \) feasibility constraint is the following
\[ \gamma C_1 \leq b \] 

and \( t = 2 \) feasibility constraint is the following

\[ (1 - \gamma)C_2 \leq Rk \]  

Central planner maximizes (1) under constraints (9), (10), and (11), and under the following incentive constraint

\[ C_1 \leq C_2 \]

FOC give us

\[
\begin{align*}
(1 - \gamma)C_2^* &= Rk^* \\
\gamma C_1^* &= b^* \\
U'(C_1^*) &= RU'(C_2^*)
\end{align*}
\]

Central Planner’s allocation is a first best allocation. Each bank keeps \( \gamma C_1^* \) consumption units invested in the short term risk free asset to face early consumer demand for liquidity. To reach optimum, at \( t = 1 \) short term funds are transferred to banks facing an excess demand for liquidity from its depositors from banks with available liquidity. At \( t = 2 \), borrowing banks refunds the lending ones by providing them long term liquidity units. Central Planner outcome exhibits an implicit rate of interest lying along the maturity transformation curve which equals \( \frac{C_2^*}{C_1^*} - 1 \).

### 2.3 Welfare comparison

When funds are transferable among regions, as there is no aggregate uncertainty about \( t = 1 \) liquidity demands, banks can store the average liquidity needs in short term asset. The available amount to be invested in the long term asset is superior when compared to autarky. Expected aggregate welfare can be written as follows.

In the autarkic case

\[
\hat{W} = \gamma U \left( \frac{1 - \bar{k}}{\lambda_h} \right) + (1 - \gamma) U \left( R \frac{\bar{k}}{1 - \gamma} \right)
\]  

In the Central Planner case

\[
W^* = \gamma U \left( \frac{1 - k^*}{\gamma} \right) + (1 - \gamma) U \left( R \frac{k^*}{1 - \gamma} \right)
\]

When fund transfers are possible, aggregate welfare is strictly higher when compared to the autarkic case because long term asset payoff is greater than one. There is thus an incentive for banks to be able to transfer funds among regions.
Early liquidation of long term assets is a way to find liquidity at \( t = 1 \). However this solution is costly and exposes the bank to the risk of undergoing a bank run. Any bank can liquidate a maximum amount of \( \beta(\lambda_h) \) units of long term asset without causing a run, (See appendix 1) with

\[
\beta(\lambda_h) = r \left( k - \frac{(1 - \lambda_h)C_1^*}{R} \right)
\]

Should an autarkic bank \( i \) have \( b* \) short term asset units, and face a high shock, \( \lambda_h \) if

\[
(\lambda_h - \gamma) C_1^* > \beta(\lambda_h)
\]

it is unable to even pay \( C_1 \) to late consumers at \( t = 2 \). There is an incentive for the latter to cheat and redeem by anticipation at \( t = 1 \). The bank is then driven to bankruptcy. This situation occurs either in case of a high shock for liquidity at \( t = 1 \), or if \( r \), the early liquidation value of capital is small enough. Conversely, should it face a low demand for liquidity, \( \lambda_l \) it would have excess short term assets when compared with the amount redeemed by its early consumers.

To tackle this liquidity misallocation, banks have an incentive in opening an interbank market so as to enable liquid institutions to lend liquidity to illiquid ones, provided that the costs of such a network (rate of interest and access cost) are not too high. The issue to solve is to know whether it is possible or not to decentralize Central Planner’s allocation through a network of interbank contacts.

3 Decentralization of first best allocation through an Interbank Network

We shall now address the question to know which exogenously given interbank network architectures makes it possible to decentralize central planner allocation. By network we shall refer to undirected graphs. The reason for the choice of undirected graph lies in the fact that a through an \( ij \) link between bank \( i \) and bank \( j \) fund can be transferred in both direction (from \( i \) to \( j \) or from \( j \) to \( i \)). Directed graphs would enable fund transfers in a single direction.

The financial network can be considered as a mean to transfer funds among regions. As considered by Allen & Gale (2000), a way to transfer funds among banks consist of a network of interbank deposits. At \( t = 0 \), banks have the opportunity to deposit part of their available funds into one or several other financial institutions. Allen & Gale’s article is based on a simple idea and studies the consequences of interbank deposits on banking stability. In their frame, liquidity shocks are randomly chosen between a high and a low value, while the amount of aggregate liquidity demand addressed to the banking system as a whole is perfectly known. To cope with illiquidity issues, banks hold cross deposits, which enable them to implement first best allocation. Their model takes into account 4 regions, in each region operates a single monopolistic bank.
They study the impact on contagion of several networks architectures when the system has to face an unexpected rise in liquidity demand. Their paper concentrate on three network structures: a complete network called complete market structure, a circular network (incomplete market structure), and a two component network (disconnected incomplete market structure). They show the stability of the networks - which they call robustness - strongly depends on the completeness of the structure of interbank claims. Complete networks are shown in their paper to be more robust to contagion than incomplete structures. Freixas, Parigi & Rochet (2000) follow a similar scheme studying different scenarios of interbank exposures through credit lines should a bank be bankrupt, or when the banking system has to face an unexpected rise in liquidity demand.

We shall turn to an alternative richer formulation. First we shall concentrate on a financial network consisting of debt contracts with interest. Second we shall consider network structures not studied by Allen & Gale, namely line network, and star shaped network. Third, we study exogenously given network architectures, with the aim of determining whether those architectures make it possible to decentralize first best allocation. There is no outside perturbation on liquidity demand. We shall then study the stability of these different structures, and their efficiency when cost are introduced.

3.1 Definitions and Assumptions on Banking Networks

We define interbank networks as follows:

**Definition 1** A link \( g_{i,j} \in \{0,1\} \) where \( g_{i,j} = 1 \) means that bank \( i \) signed a bilateral credit convention with bank \( j \). \( g_{i,j} = 0 \) means that no contract links \( i \) and \( j \). A link between banks \( i \) and \( j \) is formed if and only if \( g_{i,j} = g_{j,i} = 1 \). The resulting link is noted \( ij \).

Through these agreements liquid banks commit to lending their liquidity in excess to the requiring illiquid ones.

A network \( g = \{(g_{i,j})\} \) is a formal description of bilateral links existing among banks. Let \( G \) be the set of all non directed networks with \( 2p \) vertices.

**Definition 2** The neighborhood \( \Gamma(v) \) of a bank \( v \) is composed of the banks with which \( v \) is directly linked (not including \( v \) itself).\(^{10}\)

**Definition 3** The path length is the typical distance between every vertex \( i \) and every other vertex \( j \). It is noted \( d(i, j) \) Distance here does not refer to any defined metric space in which graph has been embedded but to a distinct graph metric, simply the number of edges (in the edge set) that must be traversed in order to reach vertex \( j \) from vertex \( i \).

The shortest path length is the minimum number of edges in the edge set that must be traversed to reach any vertex \( i \) to any other vertex \( j \). It can be noted \( \min_{i,j \in V(g)} d(i, j) \).

\(^{10}\)In technical terms we shall define the neighborhood \( \Gamma(v) \) of a vertex \( v \) as the subgraph that consists of the vertices adjacent to \( v \) (not including \( v \) itself).
Definition 4  Network $g_{2k}$ with vertex set $V(g_{2k})$ is said to be $k-$regular if any vertex $u$ within the network is linked to exactly $k$ other banks. Its degree is said to equal $k$.

$$\forall u \in V(g_{2k}), \deg(u) = k$$

A bank wants to maximize representative agent’s utility. As it is standard in models à la Diamond Dybvig, its asset / liability structure is perfectly known to each depositor. To know whether an interbank network is a way to decentralize the first best outcome, we suppose that the interbank network exists, and we shall consider exogenously given architectures in which bank $i$ realizes the following allocation:

$$(b^i, k^i) = (b^*, k^*)$$

$$(C_1^i, C_2^i) = (C_1^*, C_2^*)$$

where $(b^*, k^*, C_1^*, C_2^*)$ is central planner’s allocation.

Definition 5  A bank with less liquidity than needed is said to be illiquid. A bank with more liquidity than needed is said to be liquid or over-liquid. We shall speak of over-liquidity when a former illiquid bank receives more liquidity than needed by several of its neighbors.

A bank is said to be equilibrated when the amount of its available liquidity exactly matches the amount redeemable by its depositors at $t=1$.

Any bank acts as a compensation chamber in the following case. When an equilibrated bank has within its neighborhood an over-liquid and an illiquid bank, it acts as financial node borrowing to the over-liquid bank to lend to the illiquid one. In any other case, equilibrated bank do not communicate with their neighbors.

This interbanking economy works as follows:

$t=0$. As a first step, banks cash deposits which are identical, normalized to one unit per depositor. As a second step an exogenously given network architecture is imposed. Network structures are undirected graphs which represent the bilateral credit conventions. As a third step, banks allocate their funds.

$t=1$. As a first step, the state of nature is revealed. As a second step, the interbank lending operations are carried out, liquid banks lend to the requiring illiquid banks which they are linked with. As a third step, early consumers redeem their deposits.

$t=2$ Interbank debt is first refunded, and as a second step late depositors withdraw.

Let us precisely define the way banks behave when interbank lending operations are carried out. The communication protocol among banks within the network once the liquidity shock is known is the following. In a first round, illiquid banks signal their liquidity needs towards their neighborhood. Contacted liquid banks answer the signal by transferring the amount of available liquidity they have to one of their illiquid contact which is randomly chosen. Because
of the network structure, it may hence happen that an illiquid bank gets more funds than the amount needed to face its depositors withdrawals. This misallocation creates over-liquid nodes in the network, while some other banks remain illiquid. There is thus a need for further fund exchange. In a second round, thus, over-liquid banks signal they have excess liquidity, and illiquid institutions signals their needs towards their direct partners in the network. The same lending / borrowing process is carried out. As in first round, fund misallocation can occur i.e. banks receiving more funds than required to meet their financial obligations. The whole process is repeated as long as there are fund transfers.

Even if we can describe the process, fund exchange are supposed to be immediate, there is thus no discounting during the different rounds. Equilibrated banks do not communicate except when they are in the position of playing the role of compensation chamber.

The cost for creating a link is first to be considered to equal zero. Section 5 envisages the positive costs case. As in Aghion & alii (2000) all loans bear the same rate of interest within the network, no difference is made among debtors. This interest equals the safe interest rate \( r_1 \) plus an endogenous risk premium \( \rho \). The latter compensates for the risk associated with the banking network system. Banks are risk neutral, if we denote \( q \) as the probability for a bank in a given network to be bankrupt, we have

\[
\rho = \frac{q}{1-q} (1 + r_1)
\]

(16)

### 3.2 Decentralization of Central Planner allocation through a Common Chamber

Central planner allocation is a first best outcome. This allocation can be decentralized through a walrasian structure in which a common chamber matches all liquidity needs and excess. In such a frame banks have no bilateral link. They have just access to the common liquidity chamber which is the single liquidity provider.

**Definition 6** The Common Chamber can be defined as follows:

1) All banks are linked to a particular node which is not a bank itself to which excess liquidity is transferred at \( t = 1 \).

2) This particular node is not a bank but the market place where liquidity is exchanged. It plays no role at \( t = 0 \) or \( t = 2 \). It is just a particular path to channel liquidity at \( t = 1 \).

3) On the market place, liquidity is lent at a unique interest rate.

The common chamber is very closed to the walrasian auctioneer. Banks are in fact linked with a single clearing house which is not a player itself but is the place where liquidity supplies and demands are located. Following the second theorem of welfare, this structure can be understood as a decentralization. A Pareto Optimal network structure enables to decentralize first best outcome. As there is no aggregate uncertainty ex-ante Pareto Optimum and ex-post Pareto Optimum are identical.
Proposition 1 Common Chamber is able to decentralize Pareto Optimal allocation by charging the following factor of interest to debtors
\[ 1 + r_1 = \frac{C_2}{C_1} \]
(See Proof in Appendix 2).

This structure exhibits no bankruptcy. As there is no uncertainty on the aggregate amount of liquidity needs, all illiquid banks are able to find through the common chamber the amount they need. We can consider the common chamber as a quasi Central Bank. It is however an exclusive Central Bank. It plays indeed the role of single liquidity provider. Its neighborhood is composed of all banks within the economy, while all banks only have the common chamber in their neighborhood.

This envisaged network structure is however not played. We shall then remove the fictional Common Chamber and envisage a rather extreme case where there is no Central Bank. We are to study the possibility to get decentralized market equilibrium by direct compensation among banks through different network structures in a four region case, and then turn to a generalization.

4 Interbank Networks

4.1 The four region case

With 4 regions we ex-ante face 4 different possible networks with a single component: a line, a circular network, a complete network, or a star shaped network. Moreover we ex-ante face 4 graphs with more than one component: first a network with two line components each composed of two banks; second a network with two components one being a singleton and the other being composed of three linked banks (the three nodes component can be either a line or a triangle); third a three component network composed of two singletons and a line of two interlinked banks; and fourth a trivial empty network composed of 4 singletons.

Lemma 1 When economy is composed of 4 regions a necessary and sufficient condition for the banking network to decentralize first best is that shortest path length between two regions is at most two.

(See Proof in Appendix 3)

Lemma 1 characterize networks able to decentralize first best no bankruptcy outcome in terms of path length. However as the number of players is limited to 4, this characterization has direct consequences in term of network topology.

Proposition 2 Pareto optimal outcome with no bankruptcy may be decentralized through 3 network structures: the circular network, the complete network and the star shaped network.

Proof. It is straightforward to see that star, circular and complete network satisfy lemma 1, while all several component networks or line networks are defeated. ■
4.2 Generalization to a 2p bank network

We shall now turn to a generalization of the model considered to a structure with \( V_{2p} = \{i_1, i_2, \ldots, i_{2p}\} \) banks, which are the network vertices. The game faced by banks is similar to the previous one.

Whatever the number of banks, central planner’s allocation remains unchanged. Pareto optimal outcome is the same either in the 4 bank case or in the 2p bank case. Liquid and illiquid banks are in symmetric financial situations. The proportion of liquid banks is equal to the proportion of illiquid banks (a half each). As a consequence, there is no global shortage of liquidity in the network when bank implement central planner’s allocation. Aggregate excess liquidity equals aggregate shortage of liquidity. The issue to be solved consist in knowing first wether the network enables to decentralize Pareto optimal outcome and second wether the network enables to transfer funds among participants in a way such that illiquid institutions manage to get the liquidity they need at \( t = 1 \).

The point at stake lies thus in the way liquidity is transferred among participants though banking linkages. This issue crucially depends on the network architectures. Sticking to central planner’s allocation \((b^*, k^*, C_1^*, C_2^*)\), we shall now evaluate the capabilities of exogenously imposed network architectures to decentralize central planner’s allocation.

In the four banks case, we had to envisage 6 different liquidity shock distribution. In a 2p banks case, we face \( \binom{2p}{p} \) possible distributions. The different possible architectures are the following: all networks with several components, circular network, complete network, k-regular networks (or inter-linked regular star networks), simple star, asymmetric networks.

It is impossible to get simple topological characterization of all possible networks. That is why we shall rely on two different tools. First by a network distance characterization we shall be able to specify properties of both symmetric and asymmetric networks. Second, we shall obtain a direct topological characterization to the restricted class of symmetric networks.

**Lemma 2** A necessary and sufficient condition for a 2p bank network to decentralize first best no bankruptcy outcome is that the shortest path length between any two nodes equals at most 2.

(See appendix 4 for proof.)

Lemma 2 is just a generalization of lemma 1 to the 2p bank network case. This property enables us to characterize interbank networks in term of network distance, which includes both symmetric and asymmetric networks. It refers to the Small World property which underlines the fact that banks are separated within the interbank network only by a short chain of intermediaries. Our theoretic result on the shortest path length is validated by data. Indeed, Boss and ali (2003) show that the average path length in the undirected interbank connection network is \( 2.26 \pm 0.03 \). From those results the interbank network looks like a very Small World with about two degrees of separation and is also very close to the decentralized first best no bankruptcy outcome. Small World
networks are substitute to the walrasian common chamber system. They appear to be less strict in a sense, in so far as they rely on flexible private contracts and do not require exclusive bilateral relationships with a single liquidity provider. Thanks to this property we can draw a class of networks which is easily defeated, while another class arises as natural candidate for first best no bankruptcy outcome decentralization.

**Proposition 3** Among all possible networks, by distance characterization

i) all several components network are defeated to decentralize first best no bankruptcy outcome;  
ii) the line and the circular networks are defeated to decentralize first best no bankruptcy outcome;  
iii) star shaped networks emerges as a natural candidate to decentralize first best no-bankruptcy outcome.

(See Appendix 5 for proof)

This characterization in term of distance remains however incomplete and is thus a first approach. Even if this condition defeats easily a wide range of networks such as several component networks, too large a number of networks remain possible including irregular networks. To discriminate a step further we have to turn to a direct topological characterization. In this frame it is impossible to study all networks architecture, so we shall concentrate on the restricted class of symmetric networks. We can make the following proposition, which leads to a direct corollary.

**Proposition 4** Any $k$–regular network with $k \geq p$ decentralizes first best no-bankruptcy outcome.

(See appendix 6 for proof)

**Corollary 1** The complete network with no cost decentralizes first best no-bankruptcy outcome.

**Proof.** By definition the complete network is $(p - 1)$-regular. By previous proposition, complete network straightforwardly decentralizes Pareto Optimal outcome with no bankruptcy.  

Topological network characterization is thus a step further to discriminate among symmetric networks. There emerges a single class of networks able to decentralize first best no bankruptcy outcome. If the network is shaped in this particular way, a private system of interbank loans is able to reproduce properties of the common chamber system. Topological properties of the network may enable banks to cope with liquidity issues without turning to the fictional common chamber.

Joint to previous Small World Phenomenon, previous proposition implies that the rise in the number of agents within the network leads to the densification of the network. Total number of edges rise is indeed a quadratic function of the total number of banks. This result can easily be linked to financial liberalization.
Over the past decades, financial liberalization lead to an increase in the number of financial institutions operating on a market by two means. Firstly, on an international level, financial border liberalization has given foreign banks the opportunity of becoming domestic financial players. Secondly, on a domestic level, financial liberalization lead to the foundation of new banks, and financial corporations, even sometimes poorly funded because of the pervasiveness of new financial rules. A typical example of these two phenomena is given by South East Asian countries in the 1990’s. OECD banks opened branches all over Asia to benefit from the growth and credit boom of the ”Asian Miracle”. In the mean time, an incredible number of banks were created by local financiers. This rise in the number of banks can be interpreted in the flavor of our model as an increase in the number of players within the network. Following our results, this rise lead to a densification of financial links which are easily documented through figures from the interbank markets activity in this area. Interbank and debt markets were even created by local authorities such as the BIBF (Bangkok International Banking Facilities) to ease the financing of local banks.

5 Integration of the Access Cost

By a distance, or a topological condition, we were able to characterize networks trough which first best no bankruptcy outcome can be decentralized. However, the number of possible candidates is still large (star shaped networks and all $k$-regular networks with $k \geq p$). To discriminate further, we shall relax the no cost hypothesis. Let us note that the set of possible candidates to decentralize first best outcome with cost is a restriction on the set of possible candidates decentralizing first best without cost. Adding a link does not turn any defeated network into a possible candidate to decentralize first best.

We shall consider each bank has to pay a fixed cost $c$ for each open credit convention ($c$ is a cost per depositor). The cost is dead weight. It can stand for a cost of negotiation, the cost of accountancy documents to inform counterparts, the cost of hiring traders to manage interbank debt, or any cost to be paid to get the technology that enables fund transfers. We shall first evaluate the maximum amount a bank is ready to pay to access a network of interbank relations and then we are to study cost minimizing networks.

5.1 Costs and Optimal networks

The question here is to determine what the maximum unit cost a bank is ready to pay to partake in the network. To maximize ex-ante average consumer expected utility, each bank will partake in the financial network if and only if the later is a Pareto improvement for its clients when compared with autarky. To sign a credit convention each bank has to pay $\bar{c}$, which is the cost per depositor. Total cost paid by bank $i$ is thus a linear function of the number of conventions it signed, $n_i$.

By previous section, we know that, in the $2p$ node case, possible candidates
for first best allocation decentralization are the star shaped networks and all
$k$-regular networks with $k \geq p$. Whatever the number of participants, each bank
facing the following alternative: either it remains isolated and it has to implement
the autarkic allocation, or it participates in the network and pays the subsequent
cost depending on the number of credit conventions (links) it signs. Let us note
$(\tilde{C}_1, \tilde{C}_2, \tilde{b}, \tilde{k})$ the allocation obtained in the latter case.
The program to maximize is the following

$$\text{Max} \frac{1}{2} [\lambda_l U(C_1) + (1 - \lambda_l) U(C_2)] + \frac{1}{2} [\lambda_h U(C_1) + (1 - \lambda_h) U(C_2)]$$

(17)

Subject to the following constraints

$$b + k + n_i c \leq 1$$

$$(1 - \gamma) C_1 \leq b$$

(18)

$$C_1 < C_2$$

(19)

$$\tilde{C}_1 < C_1$$

(20)

$$\tilde{C}_2 < C_2$$

(21)

$$(1 - \gamma) \tilde{C}_2 \leq Rk$$

(22)

FOC gives us

$$\tilde{b} + \tilde{k} + n_i c = 1$$

$$\gamma \tilde{C}_1 = \tilde{b}$$

(23)

$$\tilde{C}_1 < \tilde{C}_2$$

(24)

$$\tilde{C}_1 < \tilde{C}_2$$

(25)

$$U'(\tilde{C}_1) = RU'(\tilde{C}_2)$$

(26)

plus a condition on the cost to be paid to guarantee that banks do not remain
isolated which is

$$n_i c < \frac{\lambda_h - \lambda_l}{1 - \lambda_h} (\tilde{C}_1 - 1)$$

(27)

It is thus possible to better off depositor’s situation by partaking in a network
provided the fact that the access cost remains under an upper limit given by
condition (27). We shall consider that condition (27) is verified. We have now
to study the influence of cost on network structure. It is easy to check that the
allocation with cost is Pareto inferior to the one with no cost. It’s a second best.
We have now to study network properties to implement second best alloc-
ation. The value and allocation rule in the $2p$-bank case is the same when
compared to the 4-bank case, except for the amount ”invested in the network”,
i.e. the aggregate amount paid by each bank to build links with partners which
is a linear function of the number of its links.
5.2 Cost minimizing networks

Minimizing the costs of a network has two different meanings. It can refer to the minimization of total costs incurred either by the network as a whole, or by each player. To discriminate among the latter two, we shall refer to a value function which is a way to map network costs among players. We face the following network game \((V, \mu)\) where \(V\) is the set of players (banks) and \(\mu\) is a value function on networks among those players. The network in which a bank is embedded works as a liquidity provider. Indeed, it enables each bank to reduce the amount invested in the short term asset which produces no interest. It increases on the contrary the amount invested in the long term asset. This increase in productive investment explains the rise in utility to each depositor. Moreover the network enables each embedded bank to avoid bankruptcy that happens in case of any early and costly liquidation of the long term asset when it faces a high demand for liquidity. This opportunity gain can be defined as the value of the network for the banks involved.

**Definition 7** A value function \(\mu\) is a function \(\mu : G \rightarrow \mathbb{R}\). The set of all possible value functions is noted \(\Upsilon\).

However beyond knowing how much value is generated by the interbank network, it is critical to keep track of how the value is distributed among the players. Because of the access cost paid by each network member, the value generated by the network will not be allocated according to an egalitarian rule in which any bank gets the same amount whatever its position in the network may be. It strongly depends on the number of credit conventions signed by each bank.

**Definition 8** An allocation rule is a function \(Y : G \times \Upsilon \rightarrow \mathbb{R}^{2p}\) such that \(\sum_i Y_i(\mu, g) = \mu(g)\) for all \(\mu\) and \(g\).

The allocation rule describes how the value associated with each network is distributed to the individual player. \(Y_i(\mu, g)\) is the payoff to bank \(i\) from graph \(g\) under value function \(\mu\). The allocation to each bank depends on the network structure and on the position held by the bank in the considered structure.

The alternative to any bank is the following: either it stays isolated and its depositors get the autarkic allocation \((\widehat{C}_1, \widehat{C}_2, \widehat{b}, \widehat{k})\) or it pays the access cost and the depositors get the second best allocation given by \((\widehat{C}_1, \widehat{C}_2, \widehat{b}, \widehat{k})\).

The average allocation to bank \(i\) is thus

\[
Y_i(\mu, g) = \gamma \left[ \widehat{C}_1 - \widehat{C}_1 \right] + (1 - \gamma) \left[ \widehat{C}_2 - \widehat{C}_2 \right] \tag{28}
\]

The value generated by the graph is total value to each player

\[
\mu(g) = \sum_{i \in N} Y_i(g, \mu) \tag{29}
\]
Definition 9 Egalitarian allocation rule occurs simply when all network participants get the same value from their network participation. In the $2p$ region case, egalitarian rule $Y^e$ is defined by

$$Y^e_i(g, \mu) = \frac{\mu(g)}{2p}$$

In the context of interbank network with cost, egalitarian rule occurs when each network participant pays the same total access cost to the network, i.e. when cost distribution among players is egalitarian. With our linear cost structure, this situation occurs when each vertex degree is identical.

We shall speak of minimal egalitarian cost distribution among players when this cost is minimal.

The use of value function and allocation rule enables us to exhibit a single class of networks which satisfies both second best decentralization and egalitarian cost sharing among players.

Proposition 5 Among the networks able to decentralize first best no bankruptcy outcome in the $2p$ region case, the star shaped network minimizes total cost incurred by the network. P-regular graphs are the single network structures which guarantee minimal egalitarian cost distribution among players.

(See proof in Appendix 7)

This class guarantees minimum cost incurred by each player. However, this structure is not compatible with total cost minimization; to consider this minimal cost, we have to define strong efficiency as follows.

Definition 10 A graph $g \subset G$ is strongly efficient if $\mu(g) \geq \mu(g')$ for all $g' \subset G$.

The cost structure used in the model leads to the fact that the strongly efficient network is the one that minimizes total cost paid by network participants. As a fixed cost $\bar{c}$ is paid by a bank for each credit convention it signs, the strongly efficient network is the one component minimally connected network which is the star shaped network whatever the number of network participants. However, in this case, cost structure is unbalanced, the center of the star bears a cost equal to $(2p - 1)\bar{c}$ while other participants bear a cost equal to $\bar{c}$. There is thus little chance for the star shaped network to be selected by network participants.

Proposition 6 Strong efficiency imposes a single network structure to decentralize no-bankruptcy outcome which is the star shaped network. Because of unbalanced cost, star shaped banking network should be defeated. The single credible network topology to decentralize second best no bankruptcy outcome is thus the $p-$regular network.

\[11\text{See Jackson & Wolinsky (1996)}\]
We face an intrinsic opposition between stability and efficiency. The single strongly efficient network is not stable. As a consequence, the single network topology which decentralizes second best no bankruptcy outcome is the $p$-regular network in the $2p$-region case. This network ensures minimal egalitarian cost distribution among players, but it is not efficient.

6 Conclusion

In this article, we analyze a model of interbank lending through a network of interbank relations. Bank benefit from partaking in the network when compared to the autarkic case in so far as this participation enables them to increase the amount of resources invested in the long term productive asset.

The aim of the article was to explain first why banks intend to work in networks, second how interbank networks are structured, and third to explain the densification of banking networks since the financial liberalization process that began in the 1980’s so as to give a theoretic foundation to the Small World phenomenon.

To address these issues we used a model adapted from a Diamond & Dybvig in a framework with several banks linked through a network of interbank loans. This paper is thus a first attempt to introduce concepts from network theory into banking economics. Thanks to this change, we are able to test different network structure, their influence on funds allocation, and the capability of each network to decentralize the first best no-bankruptcy outcome. We studied the influence in the rise of the number of network participants on the number of links binding banks.

In a general framework with $2p$ banks, to discriminate among all possible networks structure, we can rely on two tools: a distance characterization, or a direct topological characterization. For a network distance reason any network with several component cannot decentralize first best no bankruptcy outcome. They are not ex-ante stable as they do not enable to decentralize central planner’s allocation. Indeed, a necessary and sufficient condition for a network to decentralize first best no-bankruptcy outcome is that the shortest path length among any banks equals at most two. This condition gives thus a theoretic foundation to the idea of a Small Financial World that emerged following banking liberalization. This distance condition excludes line networks and circular network, but a large number of architectures remain possible. To discriminate further, we turned to a directly topological characterization. This one restricts analysis on symmetric networks. Among possible architectures, $k$-regular networks with $k \geq p$ decentralize for sure first best no bankruptcy outcome. This implies that a minimum density is a necessary and sufficient condition for banking networks to improve depositors welfare. The rise in the number of competing banks which characterize financial and banking liberalization thus leads to a rise in the number of links binding banks, and a rise in interbank lending operations at a rhythm more than proportional to the rise in the number of banks. Total number of edges is a quadratic function of the number of players. The higher
the number of banks, the denser are interbank lending relations. Our theoretical results seem to be comforted by stylized facts related to financial liberalization which can be characterized by a rise in interbank relations parallel to a rise in the number of financial participants.

To get a stronger characterization of network topology, we introduce linear access cost paid by banks. This enables us to highlight an opposition between strong efficiency and individual cost minimization. The star shaped network minimizes indeed the cost incurred by the network structure, but this structure leads to an asymmetric cost sharing, the center of the star bearing a cost equal to \((2p-1)c\) while any other bank pays only \(c\). \(P\)-regular network structure leads to a higher total cost but this cost is equally shared among players.

The questions that remain unsolved are the influence of this densification on interbank systemic risk. The impact of such a densification on banking systemic risk is indeed unclear. Propagation of localized financial distress can be reduced through the diversification following the densification of the network. But in turn, a larger number of links among banks may ease propagation of localized financial distress by multiplying the ways through which are transferred. This topic is to be considered by further research.

7 Appendix

7.1 Amount of long term asset a bank can liquidate without causing a run

First consider all the networks with at least one isolated bank. Consider the isolated bank, its liquid reserve is limited to \(\gamma C_1^*\). With probability a half the bank is hit by the high shock. Its liquidity shortage (amounting to \((\lambda_h - \gamma) C_1^*\)) cannot be covered by any bank loan as the bank is isolated. The single solution it has is to sell by anticipation part of its long term asset. The bank must give the late consumers at least \(C_1^*\) at date 2 otherwise the would be better off withdrawing at date 1. So a bank with a fraction \(h\) of early consumers can at most liquidate by anticipation a part \(\alpha\) of its long term asset such as

\[
R(k - \alpha) = (1 - \lambda_h)C_1^*
\]

The amount of long term asset that can be liquidated by anticipation without causing a run is thus

\[
\alpha = k - \frac{(1 - \lambda_h)C_1^*}{R}
\]

This early liquidation produces \(r\) units of liquidity at \(t = 1\). Any isolated bank can thus get by this way a maximum of \(\beta(\lambda_h)\) without causing a run, with

\[
\beta(\lambda_h) = r \left( k - \frac{(1 - \lambda_h)C_1^*}{R} \right)
\]

If

\[
(\lambda_h - \gamma) C_1^* > \beta(\lambda_h) \quad (30)
\]
then the bank experiences a run and is driven to bankruptcy. We shall con-
sider that condition (1) is satisfied. As a consequence, any isolated bank cannot
implement first best outcome, without bearing the risk of being bankrupt with
positive probability. Any isolated bank has to abide by autarkic outcome. All
networks with at least an isolated bank is thus defeated.

7.2 Proof of proposition 1

Bank \( i \) implement the following allocation :

\[
(b^i, k^i) = (b^*, k^*)
\]

\[
(C^i_1, C^i_2) = (C^*_1, C^*_2)
\]

where \((b^*, k^*, C^*_1, C^*_2)\) is central planner’s allocation.

The particular structure of the network ensures that all banks are linked
together. Total excess demand for liquidity within the network equals \( D \).

\[
D = 2 (\lambda_h C^*_1 - b^*)
\]

\[
= (\lambda_h - \lambda_l) C^*_1
\]

Total excess liquidity from liquid banks within the network equals \( L \):

\[
L = 2 (b^* - \lambda_l C^*_1)
\]

\[
= (\lambda_h - \lambda_l) C^*_1
\]

These results just highlight the fact that there is no aggregate liquidity uncer-
tainty within the network.

All banks are able to meet the demands of their depositors without liquidat-
ing long term assets while they just kept as reserves the average liquidity shock.
At \( t = 1 \), The risk for bankruptcy within the network is thus equal to zero at
\( t = 1 \).

At \( t = 2 \), transfers follows the opposite direction when compared to \( t = 1 \).
Borrowing banks pay \((1 - \lambda_h) C^*_2\) to their long term depositors, and refund
the loan they contracted plus the amount of interest, which amounts to \((1 +
\( r_1 + \rho \)) (\lambda_h - \gamma) C^*_1\). The single resource they have is the payoff from their long
term asset. On the contrary, lending banks receive their payoffs from their long
term assets plus the reimbursement from the loan they granted at \( t = 1 \). These
resources have to enable them to pay \((1 - \lambda_l) C^*_2\) to their long term consumers.

Borrowing bank budget constraint to be satisfied is thus

\[
(1 - \lambda_h) C^*_2 \leq R k^* - (1 + \rho \gamma) (\lambda_h - \gamma) C^*_1
\]

Lending bank budget constraint to be satisfied is thus

\[
(1 - \lambda_l) C^*_2 \leq R k^* + (1 + \rho \gamma) (\lambda_l - \gamma) C^*_1
\]

With

\[
1 + \rho \gamma = \frac{C^*_2}{C^*_1}
\]
constraints (17) and (18) reach saturation. The rate of interest is completely driven by the lending side of the model up to a level compatible with no bankruptcy on the borrowing side. Either at \( t = 1 \) or \( t = 2 \), there is no risk for bankruptcy. The risk premium is thus driven to zero and we have

\[ 1 + r_1 = \frac{C_2^*}{C_1^*} \]

(34)

The decentralized market interbank network is thus a way to decentralize central planner’s allocation with no bankruptcy and \( 1 + r_1 = \frac{C_2^*}{C_1^*} \). The lending / borrowing operations are compatible with the required payments for a level of interest rate on loans that equates the marginal rate of substitution of debt and the available marginal rate of transformation between short-term and long term investments.

### 7.3 Proof of Lemma 1

Proof is to be divided in two parts. First we shall prove that all networks exhibiting shortest path length strictly over 2 do not decentralize first best no bankruptcy outcome, and we shall prove that all networks with shortest path length less or equal than two decentralizes first best no bankruptcy outcome.

**Part One:**

Let us consider a four region case, in which shortest path length is strictly above 2. Let us assume bank implement first best allocation. Network structures exhibiting shortest path length strictly over 2 are the line network, and any several component networks. By convention shortest path length between two nodes within two different components in a network is infinite.

If banks implement first best allocation, the amount of available liquidity they have stored is

\[ b^* = \gamma C_1^* \]

When facing the high demand for liquidity shock the amount of liquidity redeemed by short term consumers is \( \lambda_h C_1^* \), excess demand for liquidity is thus

\[ d = (\lambda_h C_1^* - b^*) \]

Liquidity is borrowed in the network at a single rate of interest \( r_1 \) such that

\[ 1 + r_1 = \frac{C_2^*}{C_1^*} \]

By assumption banks are not able to face the excess demand by early liquidation of part of their long term asset without causing a run as

\[ (\lambda_h - \gamma) C_1^* > \beta(\lambda_h) \]

As a consequence, network composed of 4 singletons are not able to decentralize first best no bankruptcy outcome.
In the two component network, with probability $\frac{1}{3}$ one component is composed of two illiquid banks, while the other is composed of two liquid banks. The illiquid component is driven to bankruptcy.

In the line network, states $S_1$ and $S_2$ exhibit a situation in which banks are coupled along with their liquidity situation. In state $S_1$ (resp $S_2$) Bank A and B face a high (resp low) liquidity shock and banks C and D face a low (resp high) shock. In this case, the two middle banks ($B$ and $C$) exchange funds at round one with probability one as the link they share is direct, and is the single one to bind banks in different liquidity situation. Banks at the extrema (A and D) undergo no change in their respective liquidity situation at the end of round 1.

At round 2, each bank at the extrema is (A and D) is linked with a single partner which is equilibrated. In state $S_1$, bank A signals its illiquidity towards this single partner B, while bank D signals its over-liquidity toward D. However, no signal is transferred between B and C which are both equilibrated. Consequently liquidity remains located within bank D balance sheet, while A cannot cope with the liquidity withdrawals from its depositors, bank A is thus driven towards bankruptcy. In state $S_2$ the situation is similar, liquidity remains located in bank A balance sheet, while D is driven to bankruptcy.

Line network cannot decentralize first best no bankruptcy outcome.

In any case, networks with shortest path length over two are not able to decentralize first best no bankruptcy outcome.

**Part Two:**

Let us consider a four region case, in which shortest path length is less or equal than 2. Let us assume bank implement first best allocation. Network structures exhibiting shortest path length less or equal than 2 are the star shaped network, the circular network and the complete network.

If banks implement first best allocation, the amount of available liquidity they have stored is

$$b^* = \gamma C_1^*$$

When facing the high demand for liquidity shock the amount of liquidity redeemed by short term consumers is $\lambda_h C_1^*$, excess demand for liquidity is thus

$$d = (\lambda_h C_1^* - b^*)$$

Liquidity is borrowed in the network at a single rate of interest $r_1$ such that

$$1 + r_1 = \frac{C_2^*}{C_1^*}$$

By assumption banks are not able to face the excess demand by early liquidation of part of their long term asset without causing a run as

$$(\lambda_h - \gamma) C_1^* > \beta(\lambda_h)$$

In the circular network, the probability for any illiquid bank to share a common link with a liquid bank is equal to 1. In two liquidity shock distribution out of six, the probability of fund misallocation after round one is equal to a half.
In this case, after round one, there remains in the network one over-liquid bank and one illiquid bank separated one from the other by a single equilibrated bank whatever the path we follow. By compensation chamber property, the probability that funds are channeled from the over-liquid institution towards the illiquid institution at round 2 is equal to one.

As a consequence, circular network decentralizes first best no bankruptcy outcome.

In the complete network, whatever liquidity shock distribution may be, fund misallocation at first round is always possible. In this case, there remains in the network one over-liquid bank, one illiquid bank, and two equilibrated banks. Complete structure ensures that illiquid and over-liquid banks share a common direct link. Probability for the illiquid bank to be funded at round two equals thus 1.

As a consequence, complete network decentralizes first best no bankruptcy outcome.

In the star shaped network, we have to consider two case depending on the center of the star liquidity situation. Shortest path length from the center of the star to any other node equals one.

If the center of the star faces a low shock, it funds at round one with certainty one illiquid bank. There remains thus one illiquid bank and one over-liquid bank within the network indirectly connected through star center which plays the role of compensation chamber and channels the needed amount towards the illiquid institution with certainty.

If the center of the star faces a high shock, it receives with certainty all available liquidity at round one, as all other banks are only connected with the star center. As a consequence, star center is over-liquid after round one with probability one, while there remains a single illiquid institution in the network. This liquid institution shares a direct common link with the star center and is thus funded with certainty at round two.

As a consequence, complete network decentralizes first best no bankruptcy outcome.

In any case, networks with shortest path length less or equal than two are able to decentralize first best no bankruptcy outcome.

7.4 Proof of lemma 2

To prove lemma 2, we shall proceed by induction. Let $g_{2k}$ be a single component network. Let us note $V(g_{2k})$ the set of banks within $g_{2k}$.

At rank $k = 1$, lemma 1 is trivially true. To decentralize first best no bankruptcy outcome, a necessary and sufficient condition is that the two banks within the network are linked. Path length between the two network participants equals 1. At rank $k = 2$, we know from previous section that circular network decentralize first best no bankruptcy outcome. In the circular network with 4 nodes, shortest path length equals at maximum 2. So lemma 1 is true.

Let us consider that at rank $k = p$, lemma 1 is true. Network $g_{2p}$ decentralizes first best no bankruptcy outcome, and the shortest path length between
any two nodes equals at most 2.

At rank \( k = p + 1 \) two nodes \( i \) and \( j \) have been added when compared with network \( g_{2p} \). The graph \( g_{2p+2} \) can be divided in two subgraphs, \( g_{2p} \) and the subgraph with vertex set \{ \( i, j \) \}.

As
\[
\forall (u, v) \in V(g_{2p}), u \neq v, \min_{u, v \in V(g)} d(u, v) \leq 2
\]
lemma 1 is true at rank \( k = p + 1 \) if and only if
\[
\forall u \in V(g_{2p}), \forall (i, j) \in V(g_{2p+2}), (i, j) \notin V(g_{2p}) \min_{u \in V(g), l = i, j} d(u, l) \leq 2
\]

By construction, after \( t = 1 \) liquidity shock, we face a situation where \((p + 1)\) banks are illiquid while \((p + 1)\) banks are liquid.

Let us first consider \( i \) and \( j \) are liquid. Then \((p - 1)\) banks located in subgraph \( g_{2p} \) are liquid (banks in set \( V(g_{2p}) \)). There remains \((p + 1)\) banks in subgraph \( g_{2p} \) which are illiquid. Then subgraph \( g_{2p} \) exhibit an excess demand for liquidity that it cannot equilibrates through its own liquidity offer. Thus for graph \( g_{2p+2} \) to decentralize first best no bankruptcy outcome liquidity has to be transferred from \( i \) and \( j \) towards two illiquid banks in subgraph \( g_{2p} \).

Liquidity is transferred through the network either when an over-liquid bank is directly linked to an illiquid bank, or when a requiring illiquid bank and an over-liquid bank have a common neighbor which is equilibrated. In terms of network distance this implies

in case of direct link:
\[
\forall (i, j) \in V(g_{2p+2}), (i, j) \notin V(g_{2p}) \min_{u \in V(g), l = i, j} d(u, l) = 1
\]

in case of indirect fund transfer:
\[
\forall (i, j) \in V(g_{2p+2}), (i, j) \notin V(g_{2p}) \min_{u \in V(g), l = i, j} d(u, l) = 2
\]

In any case first best no bankruptcy outcome decentralization in network \( g_{2p+2} \) implies thus
\[
\forall u \in V(g_{2p}), \forall (i, j) \in V(g_{2p+2}), (i, j) \notin V(g_{2p}) \min_{u \in V(g), l = i, j} d(u, l) \leq 2.
\]

At rank \((p + 1)\) lemma 1 is true. The cases where \( i \) and \( j \) are illiquid, or where \( i \) is liquid and \( j \) illiquid are isomorphic\(^{12}\) to the case where \( i \) and \( j \) are liquid.

This implies that lemma 1 is true for all \( p > 1 \).

\(^{12}\)Isomorphic graphs can be defined as follows: Let us note \( g \) a graph with vertex set \( V \) and edge set \( E \), and \( g' \) a graph with vertex set \( V' \) and edge set \( E' \). Graphs \( g \) and \( g' \) are isomorphic if there exist a bijection \( \varphi : V \rightarrow V' \) with \( ij \in E \iff \varphi(i)\varphi(j) \in E' \) for all \( i, j \in V \). Such a map \( \varphi \) is called an isomorphism. If \( g = g' \), it is called an automorphism.
7.5 Proof of Proposition 3

Proof of i) Let us consider a network with several components. Let $c_1$ and $c_2$ be two components of the network. Let $i$ be a node within $c_1$ and $j$ be a node within $c_2$.

As there is no path between $i$ and $j$, then $d(i, j) = \infty$. (Distance between vertices in different component is infinite.)

By lemma 1, we can conclude that any graph with several components cannot decentralize first best no bankruptcy outcome.

Proof of ii) It is straightforward to see that along a line or a circular network, the shortest path length between two nodes can be superior to two.

Let $u$ and $v$ be the extrema of the line network composed of $2p$ nodes. Then the shortest path length $d(u, v) = 2p - 1 > 2$.

Let us consider a $2p$ nodes circular network. The diameter in a graph is simply the greatest distance between two elements of the vertex set. We shall note this distance $diam(u, v) = \max_{u \in V(g)} d(u, v)$. The shortest path length between two nodes lying the farther one from the other along the circular network is the diameter and equals $p > 2$.

By lemma 1, we can thus conclude that the line network and the circular network cannot decentralize first best no bankruptcy outcome.

Proof of iii) Let us consider any two banks $u$ and $v$, embedded in a $2p$ star shaped banking network. Let $h$ be the center of the star. By definition we have

1. $\forall (u, v) \in V^2(g), (u, v) \neq h, d(u, v) = 2$
2. $\forall u \in V(g), u \neq h, d(u, h) = 1$

By lemma 1, we know that the star shaped network abides by the necessary and sufficient condition to decentralize first best outcome which completes the proof.

7.6 Proof of proposition 4

We shall proceed in two parts. First let us prove that any $k$-regular network with $k = p$ decentralizes first best no bankruptcy outcome, which we call proposition 4a. Second we shall prove that any $k$-regular network with $k > p$ decentralizes first best no bankruptcy outcome, which we call proposition 4b.

We shall first proceed by induction to prove proposition 4a.

At rank $k = 2$, the 2-regular network is the circular network already studied. We know by section 4.1 that the circular network decentralizes first best no-bankruptcy outcome. Proposition 4a is true. Let $E(g_{2k})$ be the set of edges of graph $g_{2k}$.

At rank $k = p$, let us suppose $p$-regular network decentralizes first best no bankruptcy outcome. $g_{2p}$ decentralizes first best no-bankruptcy outcome. Let us show that at rank $k = (p + 1)$ the proposition is true. At rank $k = p + 1$ two nodes $i$ and $j$ have been added when compared with network $g_{2p}$. The graph $g_{2p+2}$ can be divided in two subgraph, $g_{2p}$ and the subgraph with vertex set \{i, j\}. However the number of edges has been increased by $(2p + 1)$ in the $(p + 1)$-regular network, when compared to the $p$-regular one. We have indeed:
\[ |E(g_{2p})| = p^2 \]
\[ |E(g_{2(p+1)})| = (p + 1)^2 = |E(g_{2p})| + 2p + 1. \]

As nodes \( i \) and \( j \) are embedded in a \((p + 1)\)-regular network then

\[ |\Gamma(i)| = |\Gamma(j)| = p + 1 \]

As total number of edges has increased only by \( 2p + 1 \), we can conclude that \( i \) and \( j \) are linked within \( g_{2(p+1)} \).

\( ij \subseteq g_{2(p+1)} \)

Neither loop nor multiple links are allowed by assumption. As \( g_{2(p+1)} \) is \((p + 1)\)-regular this implies that all elements of \( V(g_{2p}) \) are linked either to \( i \) or to \( j \). Bank \( i \) is directly linked to \( p \) banks, elements of \( V(g_{2p}) \); while \( j \) is directly linked to the \( p \) other banks, elements of \( V(g_{2p}) \). No nodes within \( V(g_{2p}) \) is linked both with \( i \) and \( j \). Let us note \( Sg_1 \) the subgraph composed of \( i \) and the nodes to which it is linked with, we shall alternatively note \( Sg_2 \) the subgraph composed of \( j \) and the nodes to which it is linked with.

After \( t = 1 \) liquidity shock, \((p + 1)\) banks in \( g_{2(p+1)} \) are illiquid, while \((p + 1)\) other banks are liquid. Let us suppose that \( i \) and \( j \) are liquid. Then \((p − 1)\) banks from set \( V(g_{2p}) \) are liquid. There remains \((p + 1)\) banks from set \( V(g_{2p}) \) which are illiquid. Thus for graph \( g_{2p+2} \) to decentralize first best no bankruptcy outcome liquidity has to be transferred from \( i \) and \( j \) towards two illiquid banks in subgraph \( g_{2p} \). Let us note the latter \( u \) and \( v \).

Let us first consider the case where, \( u \subseteq Sg_1 \), while \( v \subseteq Sg_2 \). As \( u \) is a element of \( Sg_1 \), we know that it has a direct link towards \( i \). As \( v \) is a element of \( Sg_2 \), we know that it has a direct link towards \( j \). Thus both \( u \) and \( v \) are to be funded by \( i \) and \( j \). No bank remain illiquid in \( V(g_{2p}) \).

Second, we have to consider the case where \( (u, v) \subseteq Sg_1 \). In this case \( u \) and \( v \) have a direct link towards \( i \). Either \( u \) or \( v \) is directly funded by \( i \). Let us consider \( u \) gets direct funding. Bank \( v \) remains unfunded. However we know \( ij \subseteq g_{2(p+1)} \) thus by compensation chamber property \( v \) is funded by \( j \) through \( i \) with probability one.

Proposition 4a is true at rank \( p + 1 \). This implies that proposition 4 is true.

Let us now prove proposition 4b: Any \( k \)-regular network with \( k > p \) decentralizes first best no-bankruptcy outcome.

We know from previous proposition that any \( k \)-regular network with \( k = p \) decentralizes first best no-bankruptcy outcome. Any bank within the \( p \)-regular network is thus able to find the liquidity amount it needs to face its financial obligations. The problem lies in the externalities stemming from the increase in the number of links.

A rise in the number of links increases the potential number of liquid links and thus the probability for an illiquid bank to be funded through the network, but it increases the number of potential illiquid links and thus the difficulty to be funded through the network.

In a \( p \)-regular network with \( 2p \) banks the probability for any illiquid bank to be funded equals one. We shall note \( \Pi' \) the probability for bank \( i \) to be funded
by a direct contact at first round. We have, with $n_i$ the number of links of bank $i$ and $n_j$ the number of links of bank $i$’s direct neighbors:

$$\Pi' = 1 - \left(1 - \frac{2}{n_j} [1 - 2^{-n_j}] \right)^{n_i}$$

(Proof in appendix 8). The complementary probability $\Pi$ is the probability for bank $i$ not to be funded by a direct contact. In a $p$-regular network with $2p$ banks, $n_i = n_i = p$ we thus have:

$$\Pi' = 1 - \left(1 - \frac{2}{p} [1 - 2^{-p}] \right)^p$$

$$\Pi = \left(1 - \frac{2}{p} [1 - 2^{-p}] \right)^p$$

As the probability to be funded at first round is $\Pi'$, the complementary probability $\Pi = 1 - \Pi'$ denotes the probability to be funded at any one of the following rounds during $t = 1$.

We have

$$\frac{\partial \Pi}{\partial p} = -\frac{\partial \Pi'}{\partial p}$$

hence, the negative externality stemming from the rise of potential illiquid partners is exactly compensated by positive externality stemming from the rise in the number of potential liquid partners. The probability to be funded for an illiquid bank remains constant equal to one if we increase each bank neighborhood size by the same number of vertices.

As a conclusion, any regular $k$-network with $k > p$ decentralizes first best no bankruptcy outcome.

### 7.7 Proof of proposition 5

(i) Let us consider the star shaped network case. Let bank A be the center of the network. Number of links is minimal in this network structure. As total cost is linear in the number of links, the star shaped network minimizes cost incurred by the network. Cost distribution is however unbalanced among players, the center of the star incurs a cost equal to $(2p - 1)\bar{c}$ while all other network participants pay only $\bar{c}$ each.

(ii) We proved that any k-regular graph with no cost with $k \geq p$ are able to decentralize first best no bankruptcy outcome. As the cost bared by any network participant is a linear function of the degree of the vertex, each bank has an interest in reducing the number of links to the minimum number that enables it to implement central planner’s allocation. This minimal number is $p$. $P$-regular graph ensures thus a minimal egalitarian cost distribution among players.
7.8 Probability to be funded by a direct contact.

Let us consider \( i \) an illiquid bank, and \( k_j \) one of its direct neighbors. \( k_j \) is liquid with probability \( \frac{1}{2} \). \( k_j \) has \( \kappa \) illiquid contacts plus \( i \). \( k_j \) lends to \( i \) with uniform probability \( \frac{1}{2} \). We have \( |\Gamma(k_j)| = n_j \). In \( k_j \)'s neighborhood there are \( n_j - (\kappa + 1) \) liquid banks, and \( (\kappa + 1) \) illiquid banks.

This situation occurs with probability \( \left( \frac{1}{2} \right)^{n_j-(\kappa+1)} \left( \frac{1}{2} \right)^{k} \). There are \( \binom{n_j-1}{\kappa} \) parts with \( \kappa \) elements among \( (n_j - 1) \) elements. The probability for \( i \) to be funded at first round by \( k_j \) direct neighbor \( \pi \) is thus equal to

\[
\pi = \sum_{\kappa=0}^{n_j-1} \left( \binom{n_j-1}{\kappa} \right) \frac{1}{\kappa+1} \left( \frac{1}{2} \right)^{n_j-(\kappa+1)} \left( \frac{1}{2} \right)^{\kappa}
\]

\[
= 2 \sum_{\kappa'=1}^{n_j} \left( \binom{n_j-1}{\kappa'} \right) \frac{1}{\kappa'} \left( \frac{1}{2} \right)^{n_j-\kappa'} \left( \frac{1}{2} \right)^{\kappa'}
\]

As

\[
\binom{n_j-1}{\kappa'} = \frac{1}{n_j} \binom{n_j}{\kappa'}
\]

we have

\[
\pi = \frac{2}{n_j} \sum_{\kappa'=1}^{n_j} \left( \binom{n_j}{\kappa'} \right) \left( \frac{1}{2} \right)^{n_j-\kappa'} \left( \frac{1}{2} \right)^{\kappa'}
\]

\[
= \frac{2}{n_j} \left( \sum_{\kappa'=0}^{n_j} \binom{n_j}{\kappa'} \left( \frac{1}{2} \right)^{n_j-\kappa'} \left( \frac{1}{2} \right)^{\kappa'} - \binom{n_j}{0} \left( \frac{1}{2} \right)^{n_j} \right)
\]

\[
= \frac{2}{n_j} \left[ 1 - 2^{n_j} \right]
\]

This implies that with probability \( (1 - \pi) \), \( j \) does not lend to \( i \) at first round. With \( |\Gamma(i)| = n_i \), no banks in \( i \)'s neighborhood lends liquidity towards \( i \) with probability

\[
\Pi = \left( 1 - \frac{2}{n_j} \left[ 1 - 2^{-n_j} \right] \right)^{n_i}
\]

Finally, \( i \) is funded at round one with complementary probability \( 1 - \Pi = \Pi' \)

\[
\Pi' = 1 - \left( 1 - \frac{2}{n_j} \left[ 1 - 2^{-n_j} \right] \right)^{n_i}
\]

References


[32] Ioannides & Datcher Loury (2002), Job Information Network, Neighborhood Effects and Inequality, mimeo: Tufts University

[34] Jackson (2003b), A New Family of Allocation Rules for Network Games, mimeo: Caltech


[42] Rochet & Tirole (1996a), Interbank Lending and Systemic Risk, Journal of Money Credit and Banking, 28(4), pp 733–762


